Universal tangle invariants and docile perturbed Gaussians

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Recent changes: 2-09 added proof of multiplicativity of the coproduct, PBW theorem for QU is for free. Lemma for existence of duals. 24-08 added graphical calculus proof of associativity of the double. The knot-like graphical calculus! 21-08 Added double construction using R-matrix and a notion of dual, no more pairing.

Abstract

We introduce a new way of computing universal knot and tangle invariants arising from ribbon Hopf algebras. Our main example comes from a quantization of $\mathfrak{sl}_2$ and provides an infinite sequence of strong yet polynomially computable knot invariants.

Contents

1 Universal invariant of tangles

1.1 v-Tangle diagrams

We work with framed oriented tangles without closed components. More precisely, a tangle consists of finitely many properly embedded intervals in $\mathbb{R}^2 \times [0,1]$ (the strands) with (distinct) endpoints on $\{0\} \times \mathbb{R} \times \{0,1\}$. Two tangles are said to be isotopic if they are related by an isotopy of $\mathbb{R}^2 \times [0,1]$ fixing the end points. By the framing we mean a choice of homotopy class of non-tangent vector fields along each strand. All tangles are considered up to isotopy and are assumed to be oriented and framed.

To describe our tangles we make use of v-tangle diagrams, an abstract notion of tangle diagram that allows greater flexibility and simplicity. Our definition is closely related to Kauffman's rotational virtual tangles [2] but we prefer not to speak about virtual crossings as they add a complication that is not even there.

Definition 1. *(v-Tangle diagram)*
A **v-tangle diagram** *is a finite oriented graph with four-valent and univalent vertices. The edges around each vertex are cyclically ordered and the vertices are marked by a sign* ± 1*. Each edge carries an orientation and an integer called the rotation number. The edges are assumed to be a disjoint union of connected oriented paths (called strands) with distinct endpoints. In addition the strands are all labeled by distinct elements of some label set. The set of v-tangle diagrams with label set* J *is called* D_J*.*

Two tangle diagrams are said to be **equivalent** *if they can be related by relabeling and the moves shown (without labels) in Figure 1.*

Applying such a Reidemeister move to a diagram D means to replace a subgraph of D that looks like the left-hand side by the tangle diagram shown in the right-hand side or vice versa.

A useful convention for drawing v-tangle diagrams is to embed any neighborhood of a vertex as a usual crossing in the plane. The sign of the vertex should coincide with the sign of the crossing. Also, the edges are immersed in the plane so that 2π times their rotation number agrees with the rotation number of the immersed interval in the plane. This however is merely a way of depicting the diagram, we insist that the actual diagram is an abstract graph, not necessarily related to the plane.

Any tangle is represented by a v-tangle diagram. To obtain a diagram we choose a projection on the plane and make sure the framing vector field is parallel to the plane. Also the tangent vector to the strands near begin/end points and all crossings points in the positive y-direction. The rotation number in the plane provides the rotation number of each edge. Of course there are also v-tangle diagrams that do not correspond to usual tangles but those are not our primary interest.

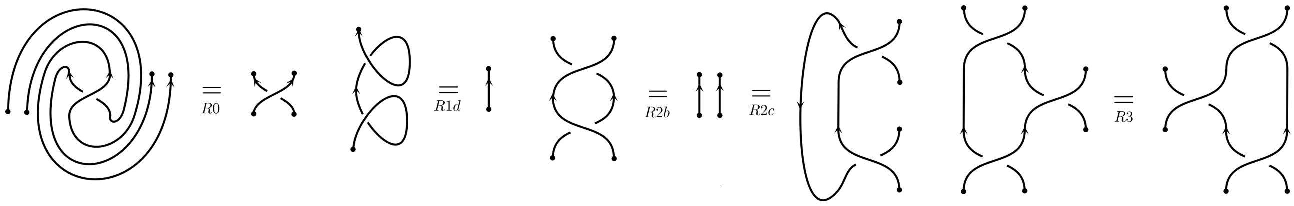


Figure 1: ACCORDING TO POLYAK WE NEED TO ADD ANOTHER CYCLIC R2 MOVE! Reidemeister moves for v-tangle diagrams, swirl $R0$, double kink $R1d$, braid-like and cyclic Reidemeister 2, $R2b$, $R2c$ and Reidemeister 3. The rotation numbers of the edges of each diagram are implied by how they are immersed in the plane.

Proposition 1. *Two v-tangle diagrams corresponding to the same tangle are equivalent, i.e. related by the moves of Figure 1.*

Proof. See [3] p.46, Thm 3.3. □

To give more precise algebraic definitions it is useful to introduce a notation for the basic labelled v-tangle diagrams, see Figure 2. First there are the positive and negative crossings $X_{ij}, \bar{X}_{ij}$ with the (cyclic) orientations as shown in the figure. The labels are i, j with the i corresponding to the overstrand if we draw X as an actual crossing and the rotation number on all edges is 0. Next $C_i, \bar{C}_i, 1_i$ represent a single crossingless strand labelled i with rotation number $1, -1, 0$ respectively. Since our v-tangle diagrams are just decorated graphs it should be clear that any v-tangle diagram can be obtained from stitching together copies of $X, \bar{X}, C, \bar{C}$ and 1.

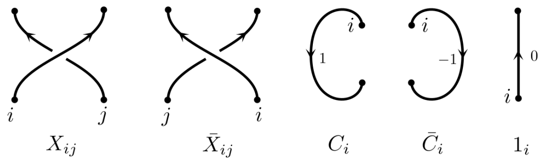


Figure 2: The basic v-tangle diagrams: Positive and negative crossing $X_{ij}, \bar{X}_{ij}$ with over-strand labelled i, the crossingless strands $C_i, \bar{C}_i$ and 1_i with rotation numbers $1, -1$ and 0.

Next let us define some operations on v-tangle diagrams called union, merging/stitching, doubling, erasing and antipoding. All these operations are natural in the sense that they extend to equivalence classes of v-tangle diagrams. As an illustration Figure 3 shows all the operations (except union) applied to the diagram D shown in the middle.

In what follows we will often use the symbol $/\!\!/$ for left-to-right composition of functions as $f /\!\!/ g = g \circ f$. As our running example to illustrate the operations we use the diagram $D \in D_{\{i,j\}}$

Definition 2. *(Operations on v-tangle diagrams)*

1. *Union. If* J, J' *are disjoint we consider the union* DD' *of* $D \in D_J$ *and* $D' \in D_{J'}$ *to be the (disjoint union) of the underlying graphs. It is an element of* $D_{J \cup J'}$*.*

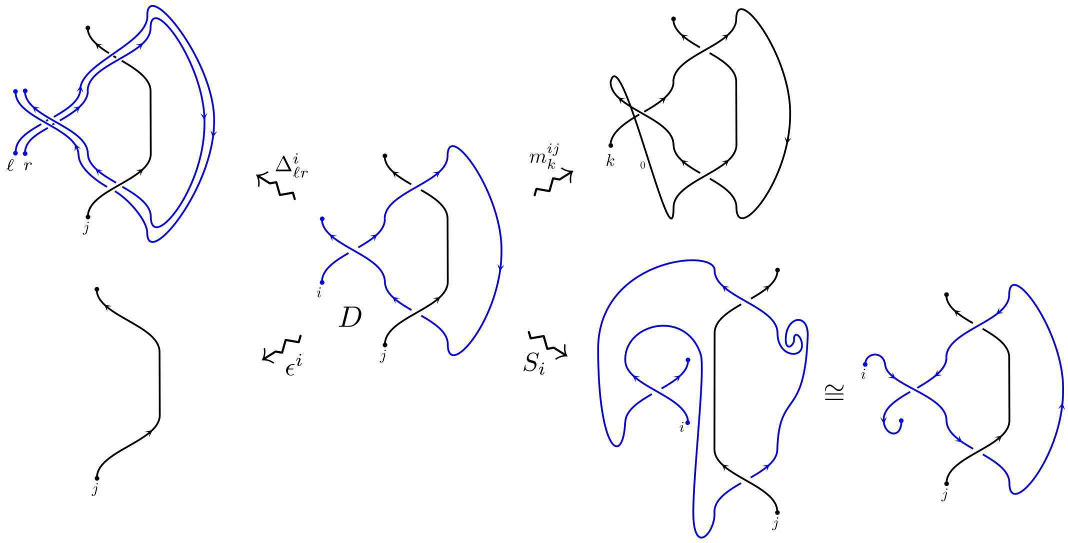
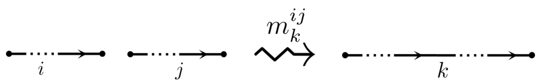


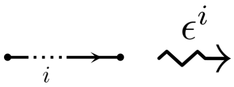
Figure 3: Applying four operations to the diagram D in the middle. The operations are merge/stitch m^i_{jk}, antipode S_i, erase ϵ^i and double $\Delta^i_{\ell r}$. For S_i we drew a second diagram that makes it more clear what is happening: the orientation of the strand i is reversed, however this is not a legal v-tangle diagram in the strict sense. Also for m^{ij}_k the edges of the graph are immersed in the plane to indicate the correct rotation numbers, the resulting graph still has 3 fourvalent vertices (crossings).

2. *Merging/stiching* m^{ij}_k.



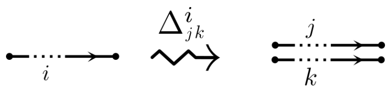
For $i, j, k \notin J$ *and* $D \in D_{\{i,j\}\cup J}$ *define* $m^{ij}_k(D) \in D_{\{k\}\cup J}$ *to be the diagram where the endpoint of strand* i *is identified with the start point of strand* j*, the resulting 2-valent vertex is removed and the rotation number on the new edge is the sum of the rotation numbers of its two parts. Finally the newly created strand is labeled* k*.*

3. *Erasing* ϵ^i.



Deletes strand labelled i*, removing all adjacent vertices.*

4. *Doubling* Δ^i_{jk} *sends a diagram* $D \in D_{\{i\}\cup J}$ *to* $D_{\{j,k\}\cup J}$*.*

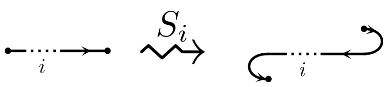


It replaces strand i *by a pair of parallel (with respect to the framing) strands called* j, k *following the same path. In terms of union, merging and the basic v-tangle diagrams doubling is defined by* $\Delta^i_{jk}(DD') = \Delta^i_{jk}(D)\Delta^i_{jk}(D')$ *and* $m^{ij}_k /\!/ \Delta^k_{\ell r} = \Delta^i_{12}\Delta^j_{34} /\!/ m^{13}_\ell /\!/ m^{24}_r$ *and* $m^{ij}_u /\!/ \Delta^k_{\ell r} = \Delta^k_{\ell r} /\!/ m^{ij}_u$ *for* $k \neq u$*, together with its effect on the basic v-tangle diagrams:*

$$X_{ij} /\!/ \Delta^i_{\ell r} = X_{\ell 2} X_{r1} /\!/ m^{12}_i \quad X_{ij} /\!/ \Delta^j_{\ell r} = X_{1\ell} X_{2r} /\!/ m^{12}_j \qquad \bar{X}_{ij} /\!/ \Delta^i_{\ell r} = \bar{X}_{\ell 1}\bar{X}_{r2} /\!/ m^{12}_i \quad \bar{X}_{ij} /\!/ \Delta^j_{\ell r} = \bar{X}_{2\ell}\bar{X}_{1r} /\!/ m^{12}_j$$

and $z_i /\!/ \Delta^i_{jk} = z_j z_k$ *for* $z \in \{C_i, \bar{C}_i, 1_i\}$ *and for any diagram* z *not containing label* i *we have* $\Delta^i_{jk}(z) = z$*.*

5. *Antipode* S_i *is a map from* $D_{\{i\}\cup J}$ *to itself*



defined by the fact that it reverses the strand i*, sending the rotation numbers of the edges of* i *to their negatives and adds a small correction to the first and last edge as shown above. In terms of generators and relations it is defined by* $S_i(DD') = S_i(D)S_i(D')$ *and* $m^{ij}_k /\!/ S_k = S_i /\!/ S_j /\!/ m^{ji}_k$ *and* $m^{ij}_u /\!/ S_k = S_k /\!/ m^{ij}_u$ *for* $k \neq u$*. On the basic v-tangle diagrams we set*

$$X_{ij} /\!/ S_j = \bar{X}_{ij} \quad X_{ij} /\!/ S_i = \bar{C}_1 \bar{X}_{2j} C_3 /\!/ m^{123}_i \quad \bar{X}_{ij} /\!/ S_i = X_{ij} \quad \bar{X}_{ij} /\!/ S_j = \bar{C}_1 \bar{X}_{i2} C_3 /\!/ m^{123}_j$$

and $1_i /\!/ S_i = 1_i, C_i /\!/ S_i = \bar{C}_i, \bar{C}_i /\!/ S_i = C_i$ *and for any diagram* z *not containing label* i *we have* $S_i(z) = z$*.*

The notation m^{123}_i means $m^{12}_j /\!/ m^{j3}_i$ or equivalently $m^{23}_j /\!/ m^{1j}_i$. More generally when merging many things we use the notation $m^{(abcd\ldots)}_i = m^{ab}_i /\!/ m^{ic}_i /\!/ m^{id}_i /\!/ \ldots$. One reason for working with our abstract notion of v-tangle diagrams is that merging is not well-defined for usual tangle diagrams, as one has to explain *how* the two strands were connected. In an abstract graph there is no such obligation. Also to save brackets our convention is that taking union always has priority so for example $AB /\!/ m^{ij}_k$ means $(AB) /\!/ m^{ij}_k$.

To further illustrate the operations on example D in Figure 3 note that in our algebraic notation

$$D = \bar{X}_{51} X_{j4} X_{26} \bar{C}_3 /\!/ m^{12345}_i /\!/ m^{j6}_j$$

According to our algebraic definition of the antipode S_i we thus get

$$D/\!\!/S_i = S_5(S_1(\bar{X}_{51}))S_4(X_{j4})S_2(X_{26})S_3(\bar{C}_3)/\!\!/m_i^{54321}/\!\!/m_j^{j6} = \bar{X}_{51}\bar{X}_{j4}\bar{C}_a\bar{X}_{b6}C_cC_3/\!\!/m_i^{543abc1}/\!\!/m_j^{j6}$$

Which is what was shown in the figure. For reference we indicated the indices of the individual pieces that were merged/stitched in Figure 4.

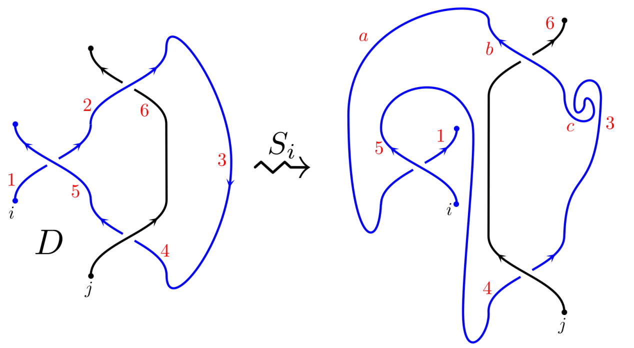


Figure 4: The v-tangle diagram D and the effect of applying antipode S to strand i both with red numbers indicating how the diagrams can be assembled from basic tangles.

Finally let us rewrite the (Reidemeister) equivalence of diagrams from Figure 1 in a purely algebraic manner using our new notation:

$$\bar{C}_1\bar{C}_2X_{34}C_5C_6/\!\!/m_i^{135}/\!\!/m_j^{246} = X_{ij} \quad (R0)$$

$$X_{13}\bar{C}_2\bar{X}_{64}\bar{C}_5/\!\!/m_i^{12345} = 1_i \quad (R1d)$$

$$\bar{X}_{12}X_{34}/\!\!/m_i^{13}/\!\!/m_j^{24} = 1_i1_j \quad (R2b)$$

$$\bar{X}_{12}X_{34}C_0/\!\!/m_i^{13}/\!\!/m_j^{402} = 1_i1_j \quad (R2c)$$

$$X_{12}X_{46}X_{53}/\!\!/m_i^{14}/\!\!/m_j^{25}/\!\!/m_k^{36} = X_{16}X_{23}X_{45}/\!\!/m_i^{14}/\!\!/m_j^{25}/\!\!/m_k^{36} \quad (R3)$$

As claimed before the above operations on v-tangle diagrams are all natural in the sense that applying an operation to both sides of one of the above equalities leads to equivalent diagrams. For Δ and S this requires some simple checking. For example we note that applying S turns $R2b$ into $R2c$. Further checks of this kind are left to the reader, see also [4].

1.2 Universal tangle invariant from a ribbon Hopf algebra

We start by establishing some notation on tensor products. For any finite set J we will use the notation $U^{\otimes J}$ for the $|J|$-th tensor power of algebra U where we label the tensor factors by the elements of the finite set J. Subscripts will be used to indicate what tensor factor an element of U belongs to and factors containing $1 \in U$ will be ommitted. For example $a_1b_2c_5$ means $a \otimes b \otimes 1 \otimes 1 \otimes c \in U^{\otimes\{1,2,3,4,5\}}$. Actually this notation is a little ambiguous as it could also denote the 3-tensor in $U^{\otimes\{1,2,5\}}$. Similarly the notation $X_{ij} \in U^{\otimes\{i,j\}}$ is used for a 2 tensor X whose first component is in tensor factor i and the second component is in factor labelled j.

The multiplication $m : U \otimes U \to U$ gives rise to maps $m_k^{ij} : U^{\otimes\{i,j\}\cup J} \to U^{\otimes\{k\}\cup J}$ where one multiplies the elements in the tensor factor labelled i by those in factor j, placing the result in factor k. In case U happens to be a Hopf algebra, the coproduct, antipode and co-unit give similar maps on the tensor powers: $\Delta^i_{jk} : U^{\otimes\{i\}\cup J} \to U^{\otimes\{j,k\}\cup J}$ and $S_i : U^{\otimes\{i\}\cup J} \to U^{\otimes\{i\}\cup J}$ and $\epsilon^i : U^{\otimes\{i\}\cup J} \to U^{\otimes J}$.

Given a ribbon Hopf algebra U with R-matrix X and ribbon element v set $C = uv^{-1}$ and $\bar{C} = uv^{-1}$ where u is the Drinfeld element. We define the universal U-invariant Z_U on tangle diagrams by decomposing the diagram into the basic tangles $X, \bar{X}, C, \bar{C}, 1$ by means of union and merging. The slogan is that merging becomes multiplication and union becomes tensor product.

Definition 3. *(Universal tangle invariant)*
Define $Z_U : D_J \to U^{\otimes J}$ on tangle diagrams as follows:

$$Z_U/\!\!/m_k^{ij} = m_k^{ij}/\!\!/Z_U \qquad Z_U(DD') = Z_U(D)Z_U(D')$$

and $Z_U(B) = B$ for all basic diagrams $B \in \{X_{ij}, \bar{X}_{ij}, C_i, \bar{C}_i, 1_i\}$.

This definition is merely an algebraic restatement of the usual definition of the universal invariant (see [3] p. 72) where one places algebra elements on the crossings, cups and caps of a tangle diagram. We do not allow closed components to ensure that the invariant always takes values in a tensor power of the algebra.

The algebraic properties of a ribbon Hopf algebra are precisely such that Z_U is well-defined and constant on equivalence classes of tangle diagrams. In other words it gives rise to an invariant of tangles. It behaves well under the natural operations that we defined above. In summary:

Theorem 1. *[3],[1]:*
Z_U is well-defined and constant on equivalence classes of tangle diagrams and hence gives rise to a tangle invariant. Furthermore, Z_U commutes with the natural operations ϵ, Δ, S:

$$Z_U /\!\!/ \epsilon^i = \epsilon^i /\!\!/ Z_U \qquad Z_U /\!\!/ S_i = S_i /\!\!/ Z_U \qquad Z_U /\!\!/ \Delta^i_{jk} = \Delta^i_{jk} /\!\!/ Z_U$$

2 Main example: The algebra $\mathbb{U}$

We now present our main example of a ribbon Hopf algebra whose universal tangle invariant we are interested in. Actually $\mathbb{U}$ is a family of topological algebras over the ring $\mathbb{Q}[\![\hbar]\!]$ depending on a parameter ϵ.

Define $\mathbb{U}$ to be h-adic completion of the $\mathbb{Q}[\![\hbar]\!]$ algebra generated by y, t, a, x subject to the relations

$$[t, .] = 0 \quad [a, x] = x \quad [a, y] = -y \quad xy - e^{\hbar\epsilon} yx = (1 - e^{-\hbar(2\epsilon a - t)})/\hbar$$

In other words the topology of $\mathbb{U}$ is that of $\mathbb{Q}[y, t, a, x][\![\hbar]\!]$.

The Hopf algebra structure is given by co-unit $\mathrm{e} : \mathbb{U} \to \mathbb{Q}[\![\hbar]\!]$ defined by $\mathrm{e}(1) = 1$ and $\mathrm{e}(g) = 0$ for any generator g and

$$\Delta(y) = y_1 + e^{\hbar(-\epsilon a_1 + t_1)} y_2 \qquad S(y) = -e^{\hbar(\epsilon a - t)} y \qquad (1)$$

$$\Delta(t) = t_1 + t_2 \qquad S(t) = -t \qquad (2)$$

$$\Delta(a) = a_1 + a_2 \qquad S(a) = -a \qquad (3)$$

$$\Delta(x) = x_1 + e^{-\hbar\epsilon a_1} x_2 \qquad S(x) = -e^{\hbar\epsilon a} x \qquad (4)$$

Notice that at $\hbar = 0$ we have the structure of a universal enveloping algebra of the Lie algebra with the same relations except now $[x, y] = 2\epsilon a - t$. All tensor products should be taken in the completed sense. The main reason for considering the $\hbar$-adic completion is that $\mathbb{U}$ the quasi-triangular with respect to the universal R-matrix

$$R_{ij} = \sum_{m,n=0}^{\infty} \frac{\hbar^{m+n} y_i^m t_i^n a_j^n x_j^m}{[m]!n!}$$

Here $q = e^{\hbar\epsilon}$ and $[m] = \frac{1-q^m}{1-q}$ and $[m]! = [1][2]\ldots[m]$ This follows directly from the fact that $\mathbb{U}$ is obtained from the Drinfeld double construction, see section XXX.

Finally $\mathbb{U}$ also has a ribbon element $v = e^{-\hbar(2\epsilon a_1 - t_1)}(X_{12} /\!\!/ S_1 /\!\!/ m_1^{12})$, see section YY We will later show that this makes $\mathbb{U}$ into a (topological) ribbon Hopf algebra. All tensor products have to be completed for this construction to make sense.

The similarity to the power series ring is strengthened by the PBW-type theorem:

Lemma 1. *As a topological algebra $\mathbb{U}$ is generated by the monomials*

$$\{y^k t^\ell a^m x^n | k, \ell, m, n \in \mathbb{N}\}$$

Proof. (prove this PBW theorem somewhere say using a variant of Chari-Pressley's ad hoc Δ argument on p.199? Or use the dequantizator.) ☐

When ϵ is invertible $\mathbb{U}$ is closely related to $\mathcal{U}_\hbar\mathfrak{sl}_2$. In fact the quotient of $\mathbb{U}$ by the two sided ideal generated by t is isomorphic to (the completion of) $\mathcal{U}_\hbar\mathfrak{sl}_2$ when ϵ is invertible. Indeed consider the map $\phi : \mathbb{U} \to \mathcal{U}_\hbar\hat{\mathfrak{sl}}_2$ defined by $\phi(a) = \frac{1}{2}H$ and $\phi(x) = Ee^{\epsilon\hbar H}$ and $\phi(y) = F$ and $\phi(t) = 0$. Then

$$e^{\epsilon\hbar H} E = \frac{1}{2}[H, e^{\epsilon\hbar H} E] = [\phi(a), \phi(x)] = \phi([a, x]) = \phi(x) = e^{\epsilon\hbar H} E$$

and (FIX THIS)

$$e^{\hbar H} E = e^{\epsilon\hbar H} EF - Fe^{\epsilon\hbar(1+H)} E = \phi(x)\phi(y) - q\phi(y)\phi(x) = \frac{1}{\hbar}(1 - e^{\epsilon\hbar H})$$

2.1 Computations in a completed PBW Hopf algebra

Consider a complete topological $\mathbb{Q}[\![\hbar]\!]$ algebra A topologically generated by the ordered sequence of generators $z = (z[1], z[2], \ldots z[g])$. In our main example $A = \mathbb{U}$ we have $z = (z[1], z[2], z[3], z[4]) = (y, t, a, x)$. We will assume the ordered monomials in z form a topological basis similar to the situation of a universal enveloping algebra of a Lie algebra. In other words we have an isomorphism of topological vector spaces $\mathbb{O} : \mathbb{Q}[z][\![\hbar]\!] \to A$ sending a commutative monomial in z to a monomial ordered according to the basis. Both elements of A and operations on them are viewed as continuous linear maps between (completed) tensor powers of A. Our objective is to describe such maps using commutative power series by pulling them back along $\mathbb{O}$.

Notice that the tensor powers of A form a category $\mathcal{A}$ as follows. The objects of $\mathcal{A}$ are finite sets and for any pair of finite sets we set $\mathcal{A}(I, J)$ to be the set of continuous linear maps from $A^{\otimes I}$ to $A^{\otimes J}$. The map $\mathbb{O}$ will give us a functor from this category to a similar category of power series that we will introduce next.

The commutative counterpart is the category $\mathcal{C}$ whose objects are finite sets and morphisms $\mathcal{C}(I, J) = \mathbb{Q}[z_j,\ j \in J][\![\hbar, \zeta_i\, i \in I]\!]$. Here $\zeta = (\zeta[1], \ldots \zeta[g])$ is a vector of variables that we think of as dual to the commutative variables z. The composition $f/\!\!/g$ of $f \in \mathcal{C}(I,J)$ and $g \in \mathcal{C}(J,K)$ is defined by $f/\!\!/g = (g|_{\zeta_j \mapsto \partial_{z_j}} f)|_{z_j=0}$. Here the index j runs over the set J. For example take $I = \{1\}, J = \{2,3\}, K = \{4,5\}$ and $f = \zeta_1 + z_3^2 + \hbar z_2 \zeta_1$ and $g = z_5 \zeta_3^2 + 2\zeta_2$. Then $f/\!\!/g = 2\hbar\zeta_1 + 2z_5$. More interestingly the identity takes the form $\mathrm{id}_I = e^{\sum_{i\in I} z_i \zeta_i}$. For instance, $z^a \zeta^b /\!\!/ e^{\sum_i \zeta_i z_i} = z^a \zeta^b$ because in pairing up the powers of z and ζ we can only pair with $\frac{1}{b!} z^b \zeta^b$ which gives precisely $z^a \zeta^b$ back again. Since the notation $(g|_{\zeta_j \mapsto \partial_{z_j}} f)|_{z_j=0}$ is a bit awkward at times we introduce the notation $\langle F \rangle_J = (F|_{\zeta_j \mapsto \partial_{z_j}})|_{z_j=0}$ where again j runs over the elements of J and the partial derivatives are assumed to be on the far left of each monomial of F.

As promised define the commutative description or differential operator functor $\mathcal{D} : \mathcal{A} \to \mathcal{C}$. On objects $\mathcal{D}$ is the identity. For finite sets I, J and a linear map $\phi \in \mathcal{A}(I,J)$ we set

$$\mathcal{D}(\phi) = (\mathbb{O}^{\otimes J})^{-1} \phi(\mathbb{O}^{\otimes I} e^{\sum_i z_i \zeta_i})$$

Lemma 2. *For finite sets I, J, K and $\phi \in \mathcal{P}(I,J), \psi \in \mathcal{P}(J,K)$ we have $\mathcal{D}(\phi/\!\!/\psi) = \mathcal{D}(\phi)/\!\!/\mathcal{D}(\psi)$. Moreover if $\mathcal{D}(\phi) = 0$ then $\phi = 0$.*

Proof.

$$\mathcal{D}(\phi/\!\!/\psi) = e^{\sum_{i\in I} z_i\zeta_i} /\!\!/ \mathbb{O}^{\otimes I} /\!\!/ \phi /\!\!/ (\mathbb{O}^{\otimes J})^{-1} /\!\!/ \mathbb{O}^{\otimes J} /\!\!/ \psi /\!\!/ (\mathbb{O}^{\otimes K})^{-1} = \mathcal{D}(\phi) /\!\!/ e^{\sum_{j\in J} z^j \zeta^j} /\!\!/ \mathbb{O}^{\otimes J} /\!\!/ \psi /\!\!/ (\mathbb{O}^{\otimes K})^{-1}$$

In the last step we used the fact that composing with $e^{\sum_{j\in J} z^j\zeta^j}$ acts as the identity in $\mathcal{C}(I,J)$ as noted in the example above. The proof of the first property is now complete.

For the second property, it follows from the fact that for any polynomial $f(z)$ we have

$$\phi(f(z)) = \phi(f(\partial_\zeta) e^{z\zeta}|_{\zeta=0}) = f(\partial_\zeta)\mathcal{D}(e^{z\zeta})|_{\zeta=0}$$

□

As an illustration we turn to the subalgebra $\mathbb{A}$ of $\mathbb{U}$ generated by a, x with relation $[x,a] = -x$. As it is the universal enveloping algebra of a Lie algebra the PBW theorem tells us the ordered monomials form a topological basis for $\mathbb{A}$. More concretely monomials can be reordered using $x^q a^r = (a-q)^r x^q$. The multiplication is a map $m_k^{ij} : \mathbb{A}^{\otimes\{i,j\}} \to \mathbb{A}^{\otimes\{k\}}$ and our goal is to find its description $\mathcal{D}(m_k^{ij}) \in \mathcal{C}(\{i,j\},\{k\}) = \mathbb{Q}[\xi_i, \xi_j, \alpha_i, \alpha_j][\![a_k, x_k, \hbar]\!]$ in terms of commutative power series in a, x and the dual variables α, ξ.

According to the definition we have

$$\mathcal{D}(m_k^{ij}) = e^{\alpha_i a_i + \alpha_j a_j + \xi_i x_i + \xi_j x_j} /\!\!/ \mathbb{O}^{\otimes\{i,j\}} /\!\!/ m_k^{ij} /\!\!/ (\mathbb{O}^{-1})^{\otimes\{k\}} = \sum_{q,r} \frac{\xi_i^q \alpha_j^r}{q!r!} (e^{\alpha_i a} x^q a^r e^{\xi_j x})_k /\!\!/ (\mathbb{O}^{-1})^{\otimes\{k\}}$$

$$= \sum_{q,r} \frac{\xi_i^q \alpha_j^r}{q!r!} (e^{\alpha_i a}(a-q)^r x^q e^{\xi_j x})_k = \sum_q \frac{\xi_i^q}{q!} (e^{(\alpha_i a + \alpha_j(a-q))} x^q e^{\xi_j x})_k = e^{(\alpha_i+\alpha_j)a_k + (e^{-\alpha_j}\xi_i + \xi_j)x_k}$$

Associativity of the algebra $\mathbb{A}$ means that $m_k^{12} /\!\!/ m_\ell^{k3} = m_k^{23} /\!\!/ m_\ell^{1k}$ and this equality can be verified by applying $\mathcal{D}$ to both sides: The left hand side becomes (using $e^{\lambda\partial_z} f(z)|_{z=0} = f(\lambda)$)

$$\mathcal{D}(m_k^{12}) /\!\!/ \mathcal{D}(m_\ell^{k3}) = \langle e^{(\alpha_1+\alpha_2)a_k + (e^{-\alpha_2}\xi_1+\xi_2)x_k + (\alpha_k+\alpha_3)a_\ell + (e^{-\hbar\alpha_3}\xi_k+\xi_3)x_\ell} \rangle_k = e^{(\alpha_1+\alpha_2+\alpha_3)a_\ell + (e^{-\alpha_2-\alpha_3}\xi_1 + e^{-\alpha_3}\xi_2+\xi_3)x_\ell}$$

and the right hand side is left as an exercise to the reader. It is also instructive to confirm that $A^{\otimes\{k\}} \ni (xa)_k = a_2 x_1 /\!\!/ \mathcal{D}(m_k^{12}) = \langle a_2 x_1 e^{(\alpha_1+\alpha_2)a_k + (e^{-\alpha_2}\xi_1+\xi_2)x_k} \rangle_{12} = \langle a_2 e^{\alpha_2 a_k} e^{-\alpha_2} x_k \rangle_2 = -x_k + a_k x_k = (a_k - 1)x_k$.

The fact that the expression summarizing multiplication in $\mathbb{A}$

$$\mathcal{D}(m_k^{ij}) = e^{(\alpha_i+\alpha_j)a_k + (e^{-\alpha_j}\xi_i+\xi_j)x_k}$$

is a relatively simple exponential formula means that multiplication in $\mathbb{A}$ may actually be simpler when carried out from this point of view. In other words we prefer manipulating exponentials of generators instead of the generators themselves. In fact we will explore in what sense the operations in our algebra send exponentials to perturbed exponentials.

2.2 $\mathcal{D}(Z_{\mathbb{U}})$ takes values in Docile Perturbed Gaussian Differential Operators

We now focus on our main example $\mathbb{U}$ where $z = (y,t,a,x)$ and $\zeta = (\eta,\tau,\alpha,\xi)$. Instead of working with general series in z, ζ we expand everything as power series in ϵ. Our main result is that in this case $\mathcal{D}$ takes all relevant tensors to a category of docile perturbed Gaussian differential operators (DoPeGDO). Let us first define this category:

Definition 4. *(***DoPeGDO***)*
Set $z = (y,t,a,x)$ and $\zeta = (\eta,\tau,\alpha,\xi)$ and define a weight wt *on these variables by* $\mathrm{wt}(y) = \mathrm{wt}(x) = \mathrm{wt}(\xi) = \mathrm{wt}(\eta) = 1$ *and* $\mathrm{wt}(\alpha) = \mathrm{wt}(t) = 0$ *and* $\mathrm{wt}(\tau) = \mathrm{wt}(a) = 2$, $\mathrm{wt}(\hbar) = 0$ *and finally* $\mathrm{wt}(\epsilon) = -4$. *Also define* $\mathcal{A} = e^\alpha, T = e^{\hbar t}$.

The category **DoPeGDO** *has finite sets as objects and the morphisms between I and J are elements of the form*

$$\omega e^{Q}(1+P) \in \mathbb{Q}[\zeta_i|i\in I][\![\hbar, z_j, j\in J]\!][\![\epsilon]\!]$$

where $\omega \in \mathbb{Q}(T^{\frac{1}{2}})$, $Q \in \mathbb{Q}(\mathcal{A}, T^{\frac{1}{2}})[z,\zeta]$ has weight 2 and the coefficients of the weight $(0,2)$ part are in $\mathbb{Q}$. Next $P = \sum_{k=1}^{\infty} P_k\epsilon^k$ with $P_k \in \mathbb{Q}(\mathcal{A}, T^{\frac{1}{2}})[z,\zeta]$ and $wt(P) \le 0$.

Composition of morphisms is defined as in $\mathcal{C}$:

$$f/\!\!/g = (g|_{\zeta_j\mapsto \partial_{z_j}} f)|_{z_j=0} \qquad j\in J$$

Theorem 2. *For any tangle K labeled by set J we have $\mathcal{D}(Z_{\mathbb{U}}(K)) \in$ **DoPeGDO**$(\emptyset, J)$. Also $\mathcal{D}(\mathrm{e}^i), \mathcal{D}(\Delta^i_{jk}), \mathcal{D}(S_i)$ and $\mathcal{D}(m^{ij}_k)$ are morphisms of* **DoPeGDO***.*

A direct consequence of docility is that the space of morphisms in **DoPeGDO** only grows slowly, i.e. as a polynomal in the complexity of the tangle diagram.

Corollary 1. *For any c-component tangle diagram D with X crossings, the invariant $\mathcal{D}(Z_{\mathbb{U}}(D))$ can be computed to order ϵ^k in $\mathcal{O}(4kXc)$ elementary ring operations in $\mathbb{Q}(\mathcal{A}, T^{\frac{1}{2}})[y,\eta,t,\tau,a,\alpha,x,\xi]$. INCORRECT FIX THIS*

Already to first order in ϵ the invariant $\mathcal{D}(Z_{\mathbb{U}}(K))$ separates the Rolfsen table of prime knots up to ten crossings. This makes our stronger than the HOMFLY polynomial and Khovanov homology[1], while they take exponential time to compute ours is computed in polynomial time.

By construction our invariant is at least as strong as the universal quantum $\mathfrak{sl}_2$ invariant or the colored Jones polynomials. In fact we expect a more precise relation between our work and the loop expansion of the colored Jones polynomial along the lines of Rozansky and Overbay.

Since our invariant is well-behaved under the natural tangle operations and is readily computable we expect it is well suited for obtaining topological applications. As a starting point we mention a close relation to the Alexander polynomial $\Delta_K(t)$:

Proposition 2.

$$\omega(K) = \mathcal{D}(Z_{\mathbb{U}}(K))|_{\epsilon=0} = \frac{1}{\Delta_K(t)}$$

An instructive way to prove this proposition is to start with any Seifert surface and use the natural tangle operations to simultaneously build the knot and its invariant (to order 0) in the guise of the Seifert formula.

Along these lines many other topological and knot theoretical phenomena may be explored. Both theoretically and practically because it is relatively easy to generate a lot of data.

The key part of the proof of our main Theorem 2 is that the R-matrix $\mathcal{D}(Z_{\mathbb{U}}(X_{ij}))$ is in **DoPeGDO**. This follows from Faddeev's formula for the q-exponential $e_q^z = \sum_n \frac{z^n}{[n]!}$:

Lemma 3. *(Faddeev, Zagier, Quesne) Recall $q = e^{\hbar\epsilon}$ and $\mathcal{D}(Z_{\mathbb{U}}(X_{ij}) = \mathbb{O}(e^{\hbar b_i a_j} e_q^{\hbar y_i x_j})$. We have*

$$e_q^z = \exp \sum_n \frac{(1-q)^n z^n}{n(1-q^n)}$$

Hence it and $\mathcal{D}(Z_{\mathbb{U}}(X_{ij})$ are in **DoPeGDO***.*

Proof. Following Zagier (add reference), since the q-exponential is equal to its q-derivative we have

$$e_q^z = \frac{e_q^{qz} - e_q^z}{qz - z} \quad \text{so} \quad \log e_q^{qz} = \log(1+(1-q)z) + \log e_q^z$$

Writing $\log e_q^z = \sum_n c_n z^n$ the previous implies that $q^n c_n = -\frac{(1-q)^n}{n} + c_n$ proving the formula. □

3 Computations in DoPeGDO

3.1 Gaussians in general

Before focusing on our main example we develop some general tools on Gaussian differential operators. We work over a field $\mathbb{F}$ of characteristic 0.

Definition 5. *Given finite sets I, J and variables $z = (z_j)_{j\in J}$ and $\zeta = (\zeta_i)_{i\in I}$, a general Gaussian is an expression*

$$e^{\zeta Q z} \sum_{k=0}^{\infty} P_k \epsilon^k \in \mathbb{F}[z_j, \zeta_j | i\in I, j\in J][\![\hbar]\!][\![\epsilon]\!]$$

Here $\zeta Q z = \sum_{i\in I, j\in J} \zeta_i Q_{ij} z_j$ for some coefficients $Q_{ij}\in\mathbb{F}$, and $P_k \in \mathbb{F}[\zeta, z]$ is of degree at most $2k$ in ζ and also at most of degree $2k$ in z.

[1]The knots 8_{16} and 10_{156} have identical HOMFLY polynomial and Khovanov Homology.

Note that the set of general Gaussians is closed under products but not sums.

Definition 6. *For any $F \in \mathbb{F}[\![\epsilon, z, \zeta]\!]$ the contraction in the variable ζ_k is defined (provided the result converges) as $\langle F\rangle_{\zeta_k} = (F|_{\zeta_k \mapsto \partial_{z_k}})|_{z_k \mapsto 0}$.*

More generally we define recursively $\langle F\rangle_{v_1,\ldots,v_g} = \langle\langle F\rangle_{v_1,\ldots v_{g-1}}\rangle_{v_g}$. Also $\langle F\rangle_K$ means contraction along all variables ζ_i for $i \in K$.

In other words to contract along ζ_k we replace the variable ζ_k by ∂_{z_k}, differentiate assuming the derivatives always are ordered first and then evaluate at $z_k = 0$. For example $\langle \zeta_1 z_2 + 2\zeta_2 z_2^2 + 3z_1 z_2 \zeta_2\rangle_{\zeta_2} = 3z_1$. When contracting several variables the end result is independent of the order of carrying out the contractions.

The crucial observation is that Gaussians are closed under contraction. This follows from the next theorem:

Theorem 3. *Suppose $P \in \mathbb{F}[\![z]\!][\zeta]$ and the labels of the variables are $I = J$ in the above and $w \in \mathbb{F}$:*

$$\langle P(z,\zeta)e^{w+\lambda\ell+\zeta l+\zeta Q z}\rangle_J = \det(\tilde{Q})e^{w+\lambda\tilde{Q}\ell}\langle P(\tilde{Q}(z+\ell), \zeta + \lambda\tilde{Q})\rangle_J$$

whenever the right hand side with $\tilde{Q} = (1-Q)^{-1}$ exists.

The key to the proof is the following lemma.

Lemma 4.

$$\langle e^{c+\lambda z + z\ell + \zeta Q z}\rangle = \det(\tilde{Q})e^{c+\lambda\tilde{Q}\ell}$$

whenever the inverse matrix $\tilde{Q} = (1-Q)^{-1}$ exists.

Proof. Without loss of generality we assume $c = 0$. The proof is to simply expand both sides explicitly and describe the resulting terms combinatorially in terms of certain labelled graphs reminiscent of Feynman diagrams.

Using the geometric series and

$$\det\tilde{Q} = \det e^{\log(1-Q)^{-1}} = e^{tr\,\log(1-Q)^{-1}} = e^{\sum_k tr\,\frac{Q^k}{k}}$$

the right hand side may be written as

$$e^{\sum_k \lambda Q^k \ell + tr\,\frac{Q^k}{k}} = e^{\sum_k \sum_{i_1\ldots i_k} \lambda_{i_0} Q_{0_1 i_1} Q_{i_1 i_2}\cdots Q_{i_{k-1}i_k}\ell_{i_k} + \frac{1}{k}Q_{i_0 i_1}Q_{i_1 i_2}\cdots Q_{i_k i_0}}$$

The left hand series can also be expanded:

$$\sum_{a_{ij},b_i,c_j=0}^{\infty}\left(\prod_{r,s\in I}\partial_{z_r}^{a_{rs}+b_r}\right)\prod_{i,j\in I}\frac{\lambda^{c_j}Q_{ij}^{a_{ij}}\ell^{b_i}}{a_{ij}!b_i!c_j!}z_j^{a_{ij}+c_j}|_{z=0}$$

Non-zero contributions are determined by a choice of the numbers a_{ij}, b_i, c_i for all $i,j \in I$ and a choice of matching each copy of ∂_r with a factor z_r for any $r \in I$. Such a choice contributes precisely $\frac{\lambda^{c_j}Q_{ij}^{a_{ij}}\ell^{b_i}}{a_{ij}!b_i!c_j!}$.

To describe both right and left hand sides more carefully we use simple Feynman type graphs that we call diagrams. A diagram is a directed graph D with only 1 and 2-valent edges and with vertices carrying labels $i \in I$. Each edge also carries a weight, for edge from i to j the weight is Q_{ij} if the vertices are both 2-valent and if i is an end-point the weight is λ_i and if j is, the weight is ℓ_j. In the latter two cases we require $i = j$. The weight $wt(D)$ of a diagram D is the product of the weights of all its edges. $\mathcal{D}$ is the set of all diagrams.

If $\mathcal{C}$ denotes the set of connected diagrams we may summarize the computation of the right hand side as:

$$RHS = e^{\sum_{G\in\mathcal{C}}\frac{wt(G)}{|Aut(G)|}}$$

$Aut(G)$ denotes the number of automorphisms of the diagram (with labelled vertices), it is of size k for a wheel graph with k edges and 1 for a path.

Next if we define an (edge) enumeration of a diagram D to be a choice of ordering the edges with a given weight. The number of edge enumerations of diagram D diagrams is called $\mathcal{E}(D)$. We claim that if $N_w(D)$ is the number of edges of weight w we have

$$LHS = \sum_{D\in\mathcal{D}} wt(D)\frac{\mathcal{E}(D)}{\prod_w N_w(D)!}$$

To explicitly see how the contributing terms are in bijection with the edge enumerated diagrams, for each i,j we arbitrarily match up a_{ij} pairs of a ∂_{z_i} with a z_j and identify each pair with an edge with beginning labeled j and end labelled i. Now matching up the edges according to how the term matches up the ∂_{z_r} with the z_r we get a diagram and after choosing arbitrary starting points in all the wheel components and an arbitrary order of the components we may enumerate each edge type according to its order of occurrence. This yields an enumerated diagram for the contributing term. Conversely, using the same conventions an enumerated diagram also produces a contributing term as we can read off precisely which ∂_r is paired with which z_r.

To finish our proof all that remains is to compute $\mathcal{D}$ in terms of the connected diagrams it consists of. We claim that

$$\mathcal{E}(D) = \frac{\prod_w N_w(D)!}{\prod_{G\in\mathcal{C}} |Aut(G)|^{n_G(D)} n_G(D)!}$$

where $n_G(D)$ is the number of copies of a connected diagram G occurring in D. This is because after ordering the edges of each weight type we overcounted by precisely reordering the connected components and also the automorphisms of each component.

Finally since $wt(D) = \prod_{G\in\mathcal{C}} wt(G)^{n_G(D)}$ we get

$$LHS = \sum_{D\in\mathcal{D}} \prod_{G\in\mathcal{C}} \frac{\left(\frac{wt(G)}{|Aut(G)|}\right)^{n_G(D)}}{n_G(D)!} = \sum_{(n_G)_{G\in\mathcal{C}}} \frac{\prod_{G\in\mathcal{C}} \left(\frac{wt(G)}{|Aut(G)|}\right)^{n_G}}{n_G!} = e^{\sum_{G\in\mathcal{C}} \frac{wt(G)}{|Aut(G)|}} = RHS$$

□

Proof. (of Theorem 3)
To derive the theorem from Lemma 4 above we introduce auxiliary variables m, μ and write

$$\langle P(z,\zeta)e^{c+\lambda z+\zeta\ell+\zeta Q z}\rangle = P(\partial_\mu, \partial_m)e^{c+(\lambda+\mu)z+\zeta(\ell+m)+\zeta Q z}|_{m=\mu=0}$$

Since these differentiations commute with contraction, replacing ℓ by $\ell + m$ and λ by $\lambda + \mu$ the lemma says

$$\langle P(z,\zeta)e^{c+\lambda z+\zeta\ell+\zeta Q z}\rangle = \det(\tilde{Q})P(\partial_\mu, \partial_m)e^{c+(\lambda+\mu)\tilde{Q}(\ell+m)}|_{m=\mu=0} =$$

$$\det(\tilde{Q})P(\tilde{Q}(\ell+m), \partial_m)e^{c+\lambda\tilde{Q}(\ell+m)}|_{m=0} = \det(\tilde{Q})e^{c+\lambda\tilde{Q}\ell}\langle P(\tilde{Q}(\ell+z), \zeta+\lambda\tilde{Q})\rangle_{z,\zeta}$$

□

From the formula in the theorem it is clear that for any docile perturbed Gaussian G its contraction $\langle G\rangle_S$ is still docile perturbed Gaussian whenever it is defined.

3.2 How to compose in DoPeGDO

Composing docile perturbed Gaussians can be done with the techiques from the previous section. For this we need two steps: first on the weight 0 or 2 variables over the field $\mathbb{Q}$ and then on the variables of weight 1 over the field $\mathbb{Q}(\mathcal{A}, T^{\frac{1}{2}})$.

3.3 Runtime estimate

4 Computations to first order

4.1 The invariant to first order in ϵ

Write down the actual tensors to first order and compute some simple examples.

4.2 Seifert formula and Alexander polynomial at $\epsilon = 0$

Say something about Burau too?

5 More on the algebra $\mathbb{U}$

5.1 A $\hbar$-adic Drinfeld double construction

Below we will construct a quasi-triangular Hopf algebra $\mathbb{D}$ from two algebras $\mathbb{A}$ and $\mathbb{B}$ and an element R in their tensor product. This is a version of the Drinfeld double construction but it is a little non-standard in the sense that we attempt to formulate everything in terms of the R-matrix R. In case $\mathbb{A}$ and $\mathbb{B}$ are finite dimensional, the reader should have no trouble recovering the usual construction by taking the dual to the R-matrix as a pairing. However in our $\hbar$-adic examples the dual to R may not exist so we prefer the following construction.

We say two elements $f \in \mathrm{Hom}(\mathbb{B}^{\otimes I} \otimes \mathbb{A}^{\otimes J}, \mathbb{B}^{\otimes K} \otimes \mathbb{A}^{\otimes L})$ and $g \in \mathrm{Hom}(\mathbb{A}^{\otimes K} \otimes \mathbb{B}^{\otimes L}, \mathbb{A}^{\otimes I} \otimes \mathbb{B}^{\otimes J})$ are duals if we have

$$(\prod_{i\in I} R_{i,\tilde{i}})(\prod_{j\in J} R_{\tilde{j}j})/\!/f/\!/(\mathrm{id}_{\mathbb{A}})_I^{\tilde{I}}(\mathrm{id}_{\mathbb{B}})_J^{\tilde{J}} = (\prod_{\ell\in L} R_{\ell,\tilde{\ell}})(\prod_{k\in K} R_{\tilde{k}k})/\!/g/\!/(\mathrm{id}_{\mathbb{A}})_L^{\tilde{L}}(\mathrm{id}_{\mathbb{B}})_K^{\tilde{K}}$$

here $\mathrm{id}_{\tilde{I}}^I$ simply renames the tensor slots in set I to the corresponding slots in $\tilde{I}$.

Suppose we have two algebras $\mathbb{A}$ and $\mathbb{B}$ and element $R \in \mathbb{B}\otimes\mathbb{A}$ with the following properties.

1. There exists a multiplicative inverse $\bar{R} \in \mathbb{B}\otimes\mathbb{A}$, so $R_{12}\bar{R}_{34}/\!/(m_{\mathbb{B}})_1^{13}(m_{\mathbb{A}})_2^{24} = 1_1 1_2$.
2. for any $f, f' \in \mathbb{A}^*$ we have $R_{21}/\!/f = R_{21}/\!/f' \Rightarrow f = f'$.

3. Also for any $g, g' \in \mathbb{B}^*$ we have $R_{12} /\!\!/ g = R_{12} /\!\!/ g' \Rightarrow g = g'$.
4. Next, assume both $m_{\mathbb{A}}$ and $m_{\mathbb{B}}$ have duals called $\Delta_{\mathbb{B}}$ and $\Delta_{\mathbb{A}}$, such that $\Delta_{\mathbb{A}}(xy) = \Delta_{\mathbb{A}}(y)\Delta_{\mathbb{A}}(y)$.
5. Also assume $1_{\mathbb{A}}$ and $1_{\mathbb{B}}$ have duals called $c_{\mathbb{B}}$ and $c_{\mathbb{A}}$.
6. Assume there exists $S_{\mathbb{A}} \in \mathrm{Hom}(\mathbb{A}, \mathbb{A})$ such that $R_{12} /\!\!/ (S_{\mathbb{A}})_2 = \bar{R}_{12}$ and that $S_{\mathbb{A}}$ has a compositional inverse $\bar{S}_{\mathbb{A}}$ and both have duals $S_{\mathbb{B}}$ and $\bar{S}_{\mathbb{B}}$ respectively.
7. Finally, assume that the element $\mu^{ij}_{k\ell} = R_{16}R_{34} /\!\!/ (\bar{S}_{\mathbb{A}})_4 /\!\!/ (m_{\mathbb{B}})^{1i3}_{k} (m_{\mathbb{A}})^{4j6}_{\ell} \in \mathrm{Hom}(\mathbb{B}^{\otimes\{i\}} \otimes \mathbb{A}^{\otimes\{j\}}, \mathbb{B}^{\otimes\{k\}} \otimes \mathbb{A}^{\otimes\{\ell\}})$ has a dual called $\tilde{\mu}^{k\ell}_{ij} \in \mathrm{Hom}(\mathbb{A}^{\otimes\{k\}} \otimes \mathbb{B}^{\otimes\{\ell\}}, \mathbb{A}^{\otimes\{i\}} \otimes \mathbb{B}^{\otimes\{j\}})$.

With all assumptions in place we are able to construct a quasi-triangular Drinfeld double $\mathbb{D} = \mathbb{B}^{cop} \otimes \mathbb{A}$.

Theorem 4. *The following defines a quasi-triangular Hopf algebra structure on the topological tensor product $\mathbb{D} = \mathbb{B}^{cop} \otimes \mathbb{A}$ with universal R-matrix R, where $\mathbb{B}^{cop}$ and $\mathbb{A}$ are Hopf sub-algebras in the obvious way. The superscript cop means the coalgebra structure is given by $\Delta_{\mathbb{D}} = \Delta^{op}_{\mathbb{B}} \otimes \Delta_{\mathbb{A}}$ and $S_{\mathbb{D}} = \bar{S}_{\mathbb{B}} \otimes S_{\mathbb{A}}$ with co-unit $c_{\mathbb{B}} \otimes c_{\mathbb{A}}$ and unit $1_{\mathbb{D}} = 1_{\mathbb{B}} \otimes 1_{\mathbb{A}}$ and finally product given by*

$$(m_{\mathbb{D}})^{i_1,i_2 j_1,j_2}_{k_1 k_2} = \tilde{\mu}^{i_2 j_1}_{i'_2, j'_1} /\!\!/ (m_{\mathbb{B}})^{i_1 j'_1}_{k_1} (m_{\mathbb{A}})^{i'_2 j_2}_{k_2}$$

Proof. The first step is to check that both $\mathbb{A}$ and $\mathbb{B}$ are Hopf algebras with the coproducts, antipodes and counits defined above.

The coassociativity of $\Delta_{\mathbb{A}}$. In other words we need to check that

$$(\Delta_{\mathbb{A}})^i_{1j} /\!\!/ (\Delta_{\mathbb{A}})^j_{23} = (\Delta_{\mathbb{A}})^i_{j3} /\!\!/ (\Delta_{\mathbb{A}})^j_{12}$$

or equivalently

$$R_{si} /\!\!/ (\Delta_{\mathbb{A}})^i_{1j} /\!\!/ (\Delta_{\mathbb{A}})^j_{23} = R_{si} /\!\!/ (\Delta_{\mathbb{A}})^i_{j3} /\!\!/ (\Delta_{\mathbb{A}})^j_{12}$$

By definition of $\Delta_{\mathbb{A}}$ as the dual to $m_{\mathbb{B}}$ we can rewrite the left hand side of the equation as:

$$R_{a1}R_{bj} /\!\!/ (m_{\mathbb{B}})^{ab}_{s} /\!\!/ (\Delta_{\mathbb{A}})^j_{23} = R_{a1}R_{b2}R_{c3} /\!\!/ (m_{\mathbb{B}})^{ab}_{s} /\!\!/ (m_{\mathbb{B}})^{bc}_{b}$$

Applying a similar argument to the right-hand side and invoking the associativity of $m_{\mathbb{B}}$ finishes the proof of coassociativity. Similarly the coassociativity of $\Delta_{\mathbb{B}}$ follows from the associativity of $m_{\mathbb{A}}$.

Next we check that $\Delta_{\mathbb{B}}$ is an algebra morphism. Starting from our assumption on $\Delta_{\mathbb{A}}$:

$$(m_{\mathbb{A}})^{ij}_{1} /\!\!/ (\Delta_{\mathbb{A}})^1_{k\ell} = (\Delta_{\mathbb{A}})^i_{12} (\Delta_{\mathbb{A}})^j_{34} /\!\!/ (m_{\mathbb{A}})^{13}_{k} (m_{\mathbb{A}})^{24}_{\ell}$$

We compose both sides with $R_{ri}R_{sj}$ and use the definition of Δ repeatedly to rewrite everything in terms of operations on $\mathbb{B}$. For the left hand side this goes as follows:

$$R_{ri}R_{sj} /\!\!/ (m_{\mathbb{A}})^{ij}_{1} /\!\!/ (\Delta_{\mathbb{A}})^1_{k\ell} = R_{ab} /\!\!/ (\Delta_{\mathbb{B}})^a_{rs} /\!\!/ (\Delta_{\mathbb{A}})^b_{k\ell} = R_{1k}R_{2\ell} /\!\!/ (\Delta_{\mathbb{B}})^a_{rs} /\!\!/ (m_{\mathbb{B}})^{12}_{a}$$

The right-hand side composed with $R_{ri}R_{sj}$ can be rewritten in a similar fashion as

$$R_{1k}R_{2\ell} /\!\!/ (\Delta_{\mathbb{B}})^1_{12} (\Delta_{\mathbb{B}})^2_{34} /\!\!/ (m_{\mathbb{B}})^{13}_{r} (m_{\mathbb{B}})^{24}_{s}$$

By the cancellation property we can get rid of the two R-matrices and finish the proof of the property.

Next we check that the antipode $S_{\mathbb{A}}$ satisfies $\Delta^i_{12} /\!\!/ (S_{\mathbb{A}})_2 /\!\!/ (m_{\mathbb{A}})^{12}_{k} = (c_{\mathbb{A}})^i /\!\!/ (1_{\mathbb{A}})_k$. Starting with the left-hand side we compose with R_{si} and expand the coproduct and the definitions of $S_{\mathbb{A}}$ and $\bar{R}$ to find:

$$R_{si} /\!\!/ \Delta^i_{12} /\!\!/ (S_{\mathbb{A}})_2 /\!\!/ (m_{\mathbb{A}})^{12}_{k} = R_{a1}R_{b2} /\!\!/ (m_{\mathbb{B}})^{ab}_{s} /\!\!/ (S_{\mathbb{A}})_2 /\!\!/ (m_{\mathbb{A}})^{12}_{k} = R_{a1}\bar{R}_{b2} /\!\!/ (m_{\mathbb{B}})^{ab}_{s} (m_{\mathbb{A}})^{12}_{k} = R_{si} /\!\!/ (c_{\mathbb{A}})^i (1_{\mathbb{A}})_k$$

The arguments for checking the other antipode axioms are analogous.

Turning to the double $\mathbb{D}$, the computations, although fairly standard, become significantly harder to follow. Given that we know that $\mathbb{A}$ and $\mathbb{B}$ are Hopf algebras we prefer to write part of our formulas graphically as follows. The R-matrix is depicted as a positive crossing pointing upwards with the over-strand in $\mathbb{B}$lue and the under-strand in $\mathbb{A}$dom which is the Hebrew word for red. Next $(m_{\mathbb{A}})^{ij}_{k}$ is interpreted as connecting the end of red strand i to the start of red strand j naming the new strand k, and similarly for $m_{\mathbb{B}}$ with blue strands. By associativity the order of carrying out the multiplications does not matter. The coproduct $(\Delta_{\mathbb{A}})^i_{\ell r}$ is interpreted as placing a trivalent vertex at the endpoints of strand i and naming the two edges that come out ℓ and r such that ℓ is to the left and r to the right with respect to the orientation on strand i. Since the coproduct is an algebra morphism and the $\Delta_{\mathbb{A}}$ is dual to $m_{\mathbb{B}}$ this amounts to doubling a blue strand. Finally the square of the antipode $(S^2_{\mathbb{A}})_i$ is interpreted appending and prepending adding full rotations on strand i, a positive rotation at the start and a negative one at the end. The interpretation for $(\bar{S}^2_{\mathbb{A}})_i$ is similar with opposite rotations. Another convention is that the blue and red strands corresponding to the same tensor factor of $\mathbb{D}$ are connected, the end of the blue to the start of the red. Of course we can only depict elements of the tensor powers of $\mathbb{D}$ that are expressible by R-matrices. By duality and the cancellation property this is sufficient for our purposes.

Before establishing associativity let us give a graphical interpretation of the multiplication $m_{\mathbb{D}}$ when it is applied to R-matrices: By definition

$$M = R_{i_1u_1}R_{u_2i_2}R_{j_1v_1}R_{v_2j_2} /\!\!/ (m_{\mathbb{D}})^{i_1 i_2, j_1 j_2}_{k_1 k_2} = R_{i_1u_1}R_{u_2i_2}R_{j_1v_1}R_{v_2j_2} /\!\!/ \tilde{\mu}^{i_2 j_1}_{i'_2 j'_1} /\!\!/ (m_{\mathbb{B}})^{i_1 j'_1}_{k_1} (m_{\mathbb{A}})^{i'_2 j_2}_{k_2}$$

By duality $R_{u_2 i_2} R_{j_1 v_1} /\!\!/ \tilde{\mu}^{i_2 j_1}_{i'_2 j'_1} = R_{x i'_2} R_{j'_1 y} /\!\!/ \mu^{xy}_{u_2 v_1}$ so that

$$M = R_{i_1 u_1} R_{x i'_2} R_{j'_1 y} R_{v_2 j_2} /\!\!/ \mu^{xy}_{u_2 v_1} /\!\!/ (m_{\mathbb{B}})^{i_1 j'_1}_{k_1} (m_{\mathbb{A}})^{i'_2 j_2}_{k_2} =$$

$$R_{i_1 u_1} R_{x i'_2} R_{j'_1 y} R_{v_2 j_2} R_{16} \bar{R}_{34} /\!\!/ (\bar{S}^2_{\mathbb{A}})_4 /\!\!/ (m_{\mathbb{B}})^{1x3}_{u_2} (m_{\mathbb{A}})^{4y6}_{v_1} /\!\!/ (m_{\mathbb{B}})^{i_1 j'_1}_{k_1} (m_{\mathbb{A}})^{i'_2 j_2}_{k_2}$$

The final formula is a formula that can be expressed graphically according to the rules explained above.

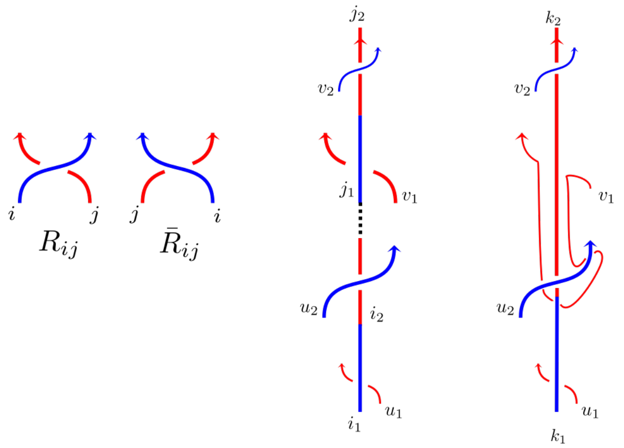


Figure 5: Crossings and a graphical interpretation of the double multiplication $m_{\mathbb{D}}$.

In this graphical language the associativity (augmented by R-matrices) now has a graphical proof given in the next figure.

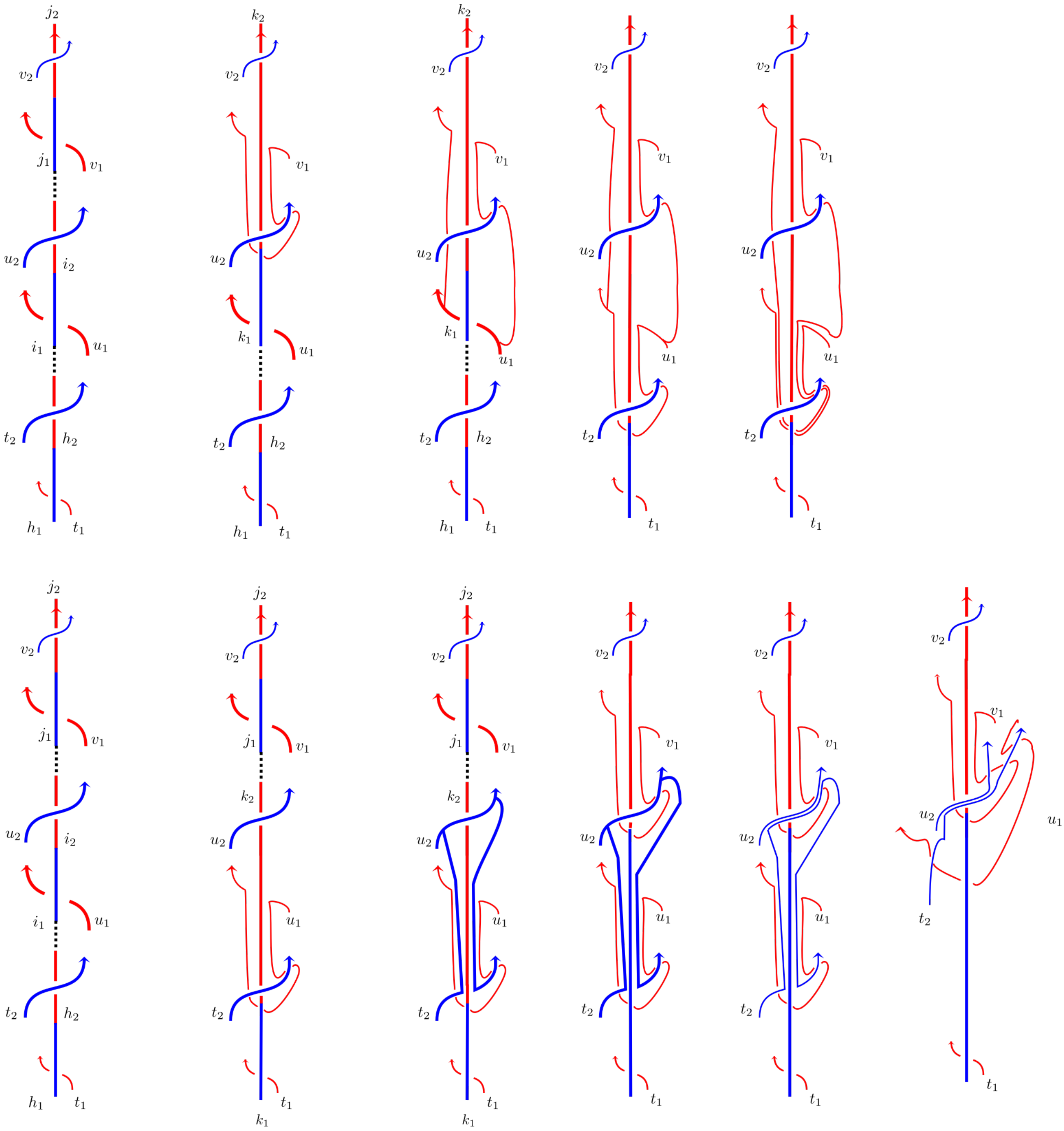


Figure 6: The graphical part of the proof of associativity of the double multiplication.

Next check that $\Delta_{\mathbb{D}}$ is an algebra morphism and check the second quasi-triangular axiom.

□

5.2 Drinfeld double construction of $\mathbb{U}$ from $\mathbb{A}$

In this section we construct the algebra $\mathbb{U}$ from $\mathbb{A}$ by the Drinfeld double construction from the previous section. Our starting point is the pair of complete $\hbar$-adic algebras

$$\mathbb{A} = \mathbb{Q}[a,x][[\hbar]]/([a,x]=x) \qquad \mathbb{B} = \mathbb{Q}[y,b][[\hbar]]/([b,y]=-\epsilon y)$$

and the element

$$R_{ij} = \sum_{m,n} \frac{\hbar^{m+n} y_i^m b_i^n a_j^n x_j^m}{[m]!n!} \in \mathbb{B}\otimes\mathbb{A}$$

To show that this input is sufficient to carry out the Drinfeld double construction as outlined in the previous section we check assumptions $1-7$. For 1 the multiplicative inverse $\bar{R}_{ij} = 1+\mathbb{O}(\hbar)$ exists and can be computed order by order in $\hbar$. We do not need the explicit formula but will require that the coefficient of

$y_i^m b_i^n$ in $\bar{R}_{ij}$ is divisible by $\hbar^{m+n}$. This follows by induction: Suppose we already found an element X_n such that $\bar{R}_{ij} = X_n + \mathbb{O}(\hbar^{n+1})$ generated by x_j, a_j and $\hbar y_i, \hbar b_i$ then we compute $X_{n+1} = X_n + Y\hbar^{n+1}$ as follows. By assumption $R_{ij}X_n = 1 + E\hbar^{n+1} + \mathbb{O}(\hbar^{n+2})$ for some $E(\hbar y_i, \hbar b_i, a_j, x_i)$. Since $R = 1 + \mathbb{O}\hbar$ we have

$$1 = R_{ij}\bar{R}_{ij} = R_{ij}(X_n + Y\hbar^{n+1} + \mathbb{O}(\hbar^{n+2})) = (E+Y)\hbar^{n+1} + \mathbb{O}(\hbar^{n+2})$$

and so we find $Y = -E$.

The cancellation properties 2 and 3 follow directly from the fact that $\frac{\hbar^{m+n}}{[m]!n!}y^m b^n$ and $a^n x^m$ are bases for $\mathbb{B}$ and $\mathbb{A}$ respectively. Any $f \in \mathbb{A}^*$ is determined by its effect on the monomials so to know $f(a_j^n x_j^m)$ we need only look at the coefficient of $\frac{\hbar^{m+n}}{[m]!n!}y_i^m b_i^n$ in $R_{ij} /\!/ f$. Similarly for $g \in \mathbb{B}^*$ the coefficient of $\frac{\hbar^{m+n}}{[m]!n!}a_i^n x_i^m$ in $R_{ji} /\!/ g$ determines $g(y_j^m b_j^n)$.

Lemma 5. *Let $\bar{\mathbb{A}}, \bar{\mathbb{B}}$ be the subalgebras generated respectively by $\hbar a, \hbar x$ and by $\hbar y, \hbar b$. The element $f \in \mathrm{Hom}(\mathbb{B}^{\otimes I} \otimes \mathbb{A}^{\otimes J}, \mathbb{B}^{\otimes K} \otimes \mathbb{A}^{\otimes L})$ has a dual $g \in \mathrm{Hom}(\mathbb{A}^{\otimes K} \otimes \mathbb{B}^{\otimes L}, \mathbb{A}^{\otimes I} \otimes \mathbb{B}^{\otimes J})$ if and only if we have*

$$(\prod_{i\in I} R_{i,\tilde{i}})(\prod_{j\in J} R_{\tilde{j}j}) /\!/ f \in \mathbb{B}^{\otimes \tilde{I}} \otimes \mathbb{A}^{\otimes \tilde{J}} \otimes \bar{\mathbb{B}}^{\otimes K} \otimes \bar{\mathbb{A}}^{\otimes L}$$

Also, if they exist, duals are unique.

Proof. The final statement follows from the cancellation properties 2, 3. □

Turning to points $4-5$ we get the existence of the dual $\Delta_{\mathbb{A}}$ directly from the lemma since $R \in \bar{\mathbb{B}} \otimes \mathbb{A}$ and $\bar{\mathbb{B}}$ is a subalgebra of $\mathbb{B}$. The existence of $\Delta_{\mathbb{B}}, c_{\mathbb{A}}, c_{\mathbb{B}}$ follows as well. To prove the algebra morphism property of $\Delta_{\mathbb{A}}$ we will compute its effect on the monomials by explicitly computing

$$R_{1i}R_{3j} /\!/ (m_{\mathbb{B}})_u^{13} = \sum_{k,\ell,m,n} \frac{\hbar^{k+\ell+m+n}}{[k]!\ell![m]!n!} y_u^{k+m}(b_u - \epsilon m)^\ell b_u^n a_i^\ell x_i^k a_j^n x_j^m = \sum_{r,s=0}^{\infty} (a_i + a_j)^s \sum_m \begin{bmatrix} r \\ m \end{bmatrix} A_i^m x_i^{r-m} x_j^m \frac{\hbar^{r+s}}{[r]!s!} y_u^r x_u^s$$

Here we set $A_i = e^{-\epsilon\hbar a_i}$. Now set up a temporary operator $\nabla : \mathbb{A} \to \mathbb{A} \otimes \mathbb{A}$ defined by $\nabla(a) = a_1 + a_2$ and $\nabla(x) = x_1 + A_1 x_2$ together with $\nabla(uv) = \nabla(u)\nabla(v)$. By induction we can show that $\Delta_{\mathbb{A}}$ and ∇ agree all monomials, not just the generators, finishing the proof of the property.

Next for point 6 we remark that $S_{\mathbb{A}}(a_1^n x_1^m)$ is the coefficient of the dual basis element $\frac{\hbar^{m+n}}{[m]!n!}y_1^m b_1^n$ in $\bar{R}_{12}$. This exists because $\bar{R} \in \bar{\mathbb{B}} \otimes \mathbb{A}$ as shown in the proof of part 1 above. The dual $S_{\mathbb{B}}$ exists as well by an application of the lemma. The compositional inverse $\bar{S}_{\mathbb{A}}$ expressed as a power series in $\mathbb{Q}[a,x][\![\alpha,\xi,\hbar]\!]$ can be computed order by order in α, ξ. Similarly for $S_{\mathbb{B}}$.

Finally for point 7 we need to show $R_{1,u}R_{v,2} /\!/ \mu_{34}^{12}$ is an element of $\mathbb{B} \otimes \mathbb{A} \otimes \bar{\mathbb{B}} \otimes \bar{\mathbb{A}}$. The lemma then implies the existence of the dual $\tilde{\mu}$. First recall that

$$R_{1,u}R_{v,2} /\!/ \mu_{34}^{12} = R_{1u}R_{v2}R_{cf}R_{de} /\!/ (\bar{S}_{\mathbb{A}})_e /\!/ (m_{\mathbb{B}})_3^{cud}(m_{\mathbb{A}})_4^{evf}$$

By the Hopf algebra properties of $\mathbb{A}$ and $\mathbb{B}$ we know that $R_{cf}R_{de} /\!/ (\bar{S}_{\mathbb{A}})_e /\!/ (m_{\mathbb{B}})_3^{cd}(m_{\mathbb{A}})_4^{ef} = (1_{\mathbb{B}})_3(1_{\mathbb{A}})_4$. Furthermore the commutation relation $x^m a^n = (a-m)^n x^m$ shows that to prove our claim we may change the order of multiplication in $m_{\mathbb{A}}$ and similarly for $m_{\mathbb{B}}$. This finishes the argument and verifies that our choice of $\mathbb{A}, \mathbb{B}, R$ satisfies the conditions 1-7.

5.3 Ribbon element and spinners/rotation

We now prove the existence of a ribbon element in $\mathbb{U}$ and prove that it takes the following form:

$$v = e^{-\hbar(2\epsilon a_1 - t_1)}(X_{12} /\!/ S_1 /\!/ m_1^{12})$$

Strategy: square roots in **DoPeGDO** are unique. First find C as a square root, then multiply C by the Drinfeld element to get v from the formula $C = uv^{-1}$.

5.4 Everything is DoPeGDO

6 Odds, ends and future directions

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7 Appendix: Computer implementation and table of knots